Math 4107, Midterm 1, Fall 2005

February 14, 2008

1. (10 points)

a. Define what it means for a set G to be a group.

b. Define what it means for a mapping φ from a group G to a group G' to be a homomorphism. Also, define what it means for φ to be an isomorphism.

c. Define what it means for a subgroup H of a group G to be a normal subgroup.

2. (20 points) Suppose that G is an abelian group of odd order.

a. Prove that the product of all the elements of G equals the identity.

b. Show that this is not true when G has even order (it is sometimes true, but not always, when G has even order) by producing a group of even order whose product of elements does not equal the identity.

3. (30 points) Suppose that G is a group of order 15. In this problem we will prove that G has an element of order 5 (in a round-about way):

Break G down into orbits under conjugation, where a and b lie in the same orbit if and only if $b = g^{-1}ag$ for some $g \in G$.

a. Conclude that if G has a non-trivial center Z, then $|Z| = 3, 5$ or 15. If $|Z| = 15$, then G is abelian, and we are done (we proved in class that abelian groups of order divisible by p always contain a p-cycle).

b. Next, show that $|Z| \neq 5$. If it did, show that any element $c \in G$, such that $c \notin Z$, has centralizer containing c and containing Z. Prove that this is not possible.

c. Next, show that if $|Z| = 3$, then there exists an element $c \in G$, $c \notin Z$, whose centralizer $C(c)$ has order 5. As in part b, do this by observing that Z and c belong to $C(c)$.

d. Show that if $|Z| = 1$, then one of the orbits of G under conjugation has order equal to 3. Conclude that the stabilizer (centralizer) of this orbit is a 5-cycle.

4. (20 points) Show that if $\varphi : G \to G'$ is an isomorphism, and if $\psi : G' \to G$ is the inverse mapping of φ, then ψ is also an isomorphism.

5. (20 points)

a. Show that $ad \equiv bd \pmod{nd}$ if and only if $a \equiv b \pmod{n}$.

b. Solve for x in the equation

$$35x \equiv 107 \pmod{9361}$$

Hint: Add multiples of 9361 to both sides and cancel off the 5 and the 7.