

A NEW PROOF OF ROTH'S THEOREM ON ARITHMETIC PROGRESSIONS

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ABSTRACT. We present a proof of Roth's theorem that follows a slightly different structure to the usual proofs, in that there is not much iteration. Although our proof works using a type of density increment argument (which is typical of most proofs of Roth's theorem), we do not pass to a progression related to the large Fourier coefficients of our set (as most other proofs of Roth do). Furthermore, in our proof, the density increment is achieved through an application of a quantitative version of Varnavides's theorem, which is perhaps unexpected.

1. INTRODUCTION

Given an integer $N \geq 1$, let $r_3(N)$ denote the size of any largest subset S of $[N] := \{1, \dots, N\}$ for which there are no solutions to

$$x + y = 2z, \quad x, y, z \in S, \quad x \neq y;$$

in other words, S has no non-trivial three term arithmetic progressions.

In the present paper we give a proof of Roth's theorem [4] that, although iterative, uses a more benign type of iteration than most proofs.

Theorem 1.1. *We have that $r_3(N) = o(N)$.*

Roughly, we achieve this by showing that $r_3(N)/N$ is asymptotically decreasing. We will do this by starting with a set $S \subseteq [N]$, $|S| = r_3(N)$, such that S has no three term progressions, and then convolving it with a measure on a carefully chosen three term arithmetic progression $\{0, x, 2x\}$. The set T where this convolution is positive will be significantly larger than S , yet will have very few three term arithmetic progressions. We will thus be able to deduce, using a quantitative version of a theorem of Varnavides [8], that $r_3(N)/N$ is much smaller than $r_3(M)/M$ for some $M = (\log N)^{1/16 - o(1)}$. It is easy to see that this implies that $r_3(N) = o(N)$. Alas, the upper bound that our method will produce for $r_3(N)$ is quite poor, and is of the quality $r_3(N) \ll N/\log_*(N)$, which nonetheless is the sort of bounds produced by the “triangle-deletion” proof of Roth's theorem [5].

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Many of the other proofs of Roth's theorem, in particular [1], [3], [6], and [7], make use of similar convolution ideas¹; however, none of these methods convolve with such a short progression as ours (three terms only), and none use the result of Varnavides to achieve a density increment. Furthermore, it seems that our method can be generalized to any context where: (1) the number of three term progressions in a set depends only on a small number of Fourier coefficients; and, (2) one has a quantitative version of Varnavides's theorem. This might prove especially useful in certain contexts, because the particular sets on which our method achieves a density increment (via Varnavides) are unrelated to the particular additive characters where the Fourier transform of S is "large"².

2. NOTATION

We shall require a modicum of notation: given a function $f : \mathbb{F}_p \rightarrow [0, 1]$, we write

$$\Lambda(f) := \mathbb{E}_{x,d \in \mathbb{F}_p} f(x)f(x+d)f(x+2d)$$

(where \mathbb{E} represents an averaged sum; thus the \mathbb{E} above represents $p^{-2} \sum$). Thus Λ gives an average of f over three term arithmetic progressions; when f is the indicator function of a set A , this is just the number of progressions in A divided by p^2 . We shall make use of the Fourier transform $\widehat{f} : \mathbb{F}_p \rightarrow \mathbb{C}$ of a function f , given by

$$\widehat{f}(r) := \mathbb{E}_{x \in \mathbb{F}_p} f(x)e^{2\pi irx/p},$$

as well as the easily-verified Parseval's identity

$$\sum_{r \in \mathbb{F}_p} |\widehat{f}(r)|^2 = \mathbb{E}_x |f(x)|^2.$$

It is also easy to check that

$$\Lambda(f) = \sum_{r \in \mathbb{F}_p} \widehat{f}(r)^2 \widehat{f}(-2r). \quad (2.1)$$

Given a set $T \subseteq \mathbb{F}_p$, we shall furthermore use the notation

$$\Lambda(T) := \Lambda(1_T).$$

Finally, the notation $\|t\|_{\mathbb{T}}$ will be used to denote the distance from t to the nearest integer.

¹In the case of Szemerédi's argument [7], the convolution is disguised, but after the dust has settled, one will see that he convolves with a measure on a very long arithmetic progression. In the case of [3] and [6], the arguments can be directly expressed in terms of convolution with a measure supported on a long arithmetic progression.

²That is, the progression to which we pass with each iteration is unrelated to the additive characters where $\widehat{1_S}$ is "large".

3. PROOF OF THEOREM 1.1

Let

$$\kappa := \limsup_{N \rightarrow \infty} r_3(N)/N.$$

We shall show that $\kappa = 0$, which will prove the theorem.

Let $N \geq 2$ be an integer, and then let p be a prime number satisfying

$$2N < p < 4N.$$

The fact that such a p exists is of course the content of Bertrand's postulate.

Let $S \subset [N]$ be a set free of three term progressions with $|S| = r_3(N)$. Thinking of S as a subset of \mathbb{F}_p in the obvious way, we shall write $f = 1_S : \mathbb{F}_p \rightarrow \{0, 1\}$ for the indicator function of S . Let

$$R := \{r \in \mathbb{F}_p : |\widehat{f}(r)| \geq (2 \log \log p / \log p)^{1/2}\}.$$

By Parseval's identity, this set of large Fourier coefficients cannot be too big; certainly,

$$|R| \leq \log p / 2 \log \log p.$$

We may therefore dilate these points of R to be contained in a short part of \mathbb{F}_p . Indeed, by Dirichlet's box principle there is an integer dilate x satisfying

$$0 < x < p^{1-1/(|R|+1)} \leq p / \log p,$$

such that for all $r \in R$ we have

$$\|xr/p\|_{\mathbb{T}} \leq p^{-1/(|R|+1)} \leq 1 / \log p. \quad (3.1)$$

Taking such an x , define

$$B := \{0, x, 2x\},$$

and define h to be the normalised indicator function for B , given by

$$h(n) := p1_B(n)/3.$$

Then convolve f with h to produce the new function

$$g(n) := (f * h)(n) = (f(n) + f(n-x) + f(n-2x))/3.$$

Since

$$\widehat{f}(r) - \widehat{g}(r) = \widehat{f}(r)(1 - \widehat{h}(r)),$$

it is easy to check using (3.1) that for all $r \in \mathbb{F}_p$

$$|\widehat{f}(r) - \widehat{g}(r)| \ll (\log \log p / \log p)^{1/2}.$$

From this, along with the Cauchy-Schwarz inequality, Parseval's identity, and equation (2.1), one can quickly deduce that

$$|\Lambda(f) - \Lambda(g)| \ll (\log \log p / \log p)^{1/2},$$

and therefore since $\Lambda(f) \ll 1/p$ (because S is free of three term arithmetic progressions), we deduce

$$\Lambda(g) \ll (\log \log p / \log p)^{1/2}. \quad (3.2)$$

Define

$$T := \{n \in \mathbb{F}_p : g(n) > 0\},$$

and note that from (3.2), along with the obvious fact that $\Lambda(T) \ll \Lambda(g)$, we have

$$\Lambda(T) \ll (\log \log p / \log p)^{1/2}. \quad (3.3)$$

Furthermore, since S is free of three term progressions even in \mathbb{F}_p , we must have that $g(n) \leq 2/3$ for all $n \in \mathbb{F}_p$. Thus $1_T(n) \geq 3g(n)/2$ for all n , immediately implying that $|T| \geq 3|S|/2$. The set T would thus serve our purposes if it was not for the fact that it is not necessarily contained in $[N]$. However, since $x \leq p/\log p$, we certainly have the inclusion $T \subset [N + 2p/\log p]$. So, if we let T' be those elements of T lying in $[N]$, then

$$|T'| = |T| - O(N/\log N) \text{ and } \Lambda(T') \leq \Lambda(T).$$

Hence, for N large enough,

$$|T'| \geq 4|S|/3$$

(unless of course $r_3(N) = O(N/\log N)$, but then we would be happy anyway).

We have now created a set T' , significantly larger than S , but with only a few more three term progressions. The following lemma, a quantitative version of Varnavides's theorem, will help us make use of this information. The notation $T_3(X)$ denotes the number of three term progressions $a, a + d, a + 2d$ with $d \geq 1$ in a set X of integers.

Lemma 3.1. *For any $1 \leq M \leq N$, and for any set $A \subseteq [N]$, we have*

$$T_3(A) \geq \left(\frac{|A|/N - (r_3(M) + 1)/M}{M^4} \right) N^2.$$

Before we prove this, let us see how we can use it to finish the proof of our main theorem. Set $M := \lfloor (\log p / \log \log p)^{1/16} \rfloor$ and apply the lemma to our set T' to obtain the estimate

$$\Lambda(T') \gg \frac{4|S|/3N - (r_3(M) + 1)/M}{M^4}.$$

Comparing this to (3.3) (recalling that $\Lambda(T') \leq \Lambda(T)$), we conclude that

$$r_3(N)/N = |S|/N \leq 3r_3(M)/4M + O((\log \log N / \log N)^{1/4}).$$

Thus $r_3(N)/N$ is asymptotically decreasing to 0, whence $\kappa = 0$.

Proof of Lemma 3.1. The result will follow from an averaging procedure essentially contained in [2]. We include the proof here since our formulation is slightly different: we are working over $[N]$ rather than \mathbb{F}_p , and so we have to take into account the inhomogeneity of $[N]$.

Let k be a positive integer. Let \mathcal{B} denote the collection of length M arithmetic progressions contained in $[N]$ with common difference at most k , and let \mathcal{B}_d denote the subcollection consisting of such arithmetic progressions with common difference d . Throughout this proof we restrict ourselves to progressions with positive common difference.

We first claim that any 3AP (three term progression) in $[N]$ can occur in at most $M^2/4$ progressions in \mathcal{B} . To see this, note that if a 3AP has common difference d , then it can occur in at most $M - 2$ progressions of length M with common difference d . Similarly,

the 3AP can occur in at most $M - 2d/n$ M -APs with difference n provided n divides d and $n \geq 2d/(M - 1)$, and in no other M -APs. Thus the 3AP can occur in no more than

$$\sum_{1 \leq m \leq (M-1)/2} (M - 2m) \leq M^2/4$$

members of \mathcal{B} , as claimed. It follows immediately that

$$T_3(A) \geq \frac{4}{M^2} \sum_{B \in \mathcal{B}} T_3(A \cap B). \tag{3.4}$$

Now if B is an arithmetic progression of length M and $|A \cap B| > r_3(M)$, then by definition we have $T_3(A \cap B) \geq 1$. In view of (3.4) our aim shall therefore be to estimate the number of such sets B ; we shall do this by looking at progressions of fixed common differences. Indeed, for a fixed common difference d , every element in the interval $I_d := [(M - 1)d + 1, N - (M - 1)d]$ is contained in precisely M progressions in \mathcal{B}_d , and so

$$\sum_{B \in \mathcal{B}} |A \cap B| = \sum_{d \leq k} \sum_{a \in A} \sum_{B \in \mathcal{B}_d} 1_B(a) \geq M \sum_{d \leq k} |A \cap I_d|.$$

Since $|A \cap I_d| \geq |A| - 2(M - 1)d$, this quantity is at least $Mk(|A| - 2Mk)$. Now let $\mathcal{C} \subset \mathcal{B}$ be the set of progressions B for which $|A \cap B| > r_3(M)$. We then have

$$\sum_{B \in \mathcal{B}} |A \cap B| \leq M|\mathcal{C}| + r_3(M)|\mathcal{B} \setminus \mathcal{C}|,$$

from which it follows that

$$|\mathcal{C}| \geq k(|A| - 2Mk) - |\mathcal{B}|r_3(M)/M.$$

Since $|\mathcal{B}_d| = N - (M - 1)d$ for each d , the total number of progressions $|\mathcal{B}|$ is at most Nk . Choosing $k = \lfloor N/2M^2 \rfloor$ we conclude that there must be at least

$$|\mathcal{C}| \geq \left(\frac{|A|/N - r_3(M)/M - 1/M}{4M^2} \right) N^2$$

sets B for which $|A \cap B| > r_3(M)$. The result thus follows from (3.4). □

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