The Rayleigh Principle for Finding Eigenvalues

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1 Introduction

Here I will explain how to use the Rayleigh principle to find the eigenvalues of a matrix A. Recall that given a symmetric, positive definite matrix A we define

$$R(x) \ = \ \frac{x^T Ax}{x^T x}.$$

Here, the numerator and denominator are 1 by 1 matrices, which we interpret as numbers.

2 Scaling

The first principle for finding $R(x)$ that I want to mention is scaling. Basically, what I mean is that as far as minimizing (or maximizing) $R(x)$ is concerned, we can restrict ourselves to x's such that $||x|| = 1$; for example,

$$\min_x R(x) \ = \ \min_{||x||=1} R(x).$$

Why is this so? Well, suppose that x is any vector which minimizes $R(x)$, say $x = (x_1, ..., x_n)$, and let $c^2 = x_1^2 + \cdots + x_n^2 = ||x||^2$. Then, consider the vector

$$y \ = \ (y_1, ..., y_n) \ = \ (x_1/c, x_2/c, ..., x_n/c) \ = \ x/c.$$

Then,

$$||y||^2 \ = \ \frac{x_1^2 + \cdots + x_n^2}{c^2} \ = \ 1.$$

Also, note that

$$R(x) \;=\; \frac{x^TAx}{x^Tx} \;=\; \frac{(1/c^2)x^TAx}{(1/c^2)x^Tx} \;=\; \frac{(x/c)^TA(x/c)}{(x/c)^T(x/c)} \;=\; \frac{y^TAy}{y^Ty} \;=\; R(y).$$

We may likewise restrict ourselves to vectors y which satisfy any constraint such as

$$a_1y_1^2 + \cdots + a_ny_n^2 \;=\; c \;>\; 0.$$

By that I mean the following:

$$\min_x R(x) \;=\; \min_{\substack{y=(y_1,\ldots,y_n)\\ a_1y_1^2+\cdots+a_ny_n^2=c}} R(y).$$

3 The values of $R(x)$

Since A is symmetric we know that it has all real eigenvalues

$$\lambda_1 \;\leq\; \lambda_2 \;\leq\; \cdots \;\leq\; \lambda_n.$$

What are the possible values that $R(x)$ can take on in terms of these eigenvalues? Well, we first observe that since A is symmetric we know that

$$A \;=\; Q^T\Lambda Q,$$

where Λ is diagonal and Q is orthogonal. So,

$$R(x) \;=\; \frac{x^TAx}{x^Tx} \;=\; \frac{(Qx)^T\Lambda(Qx)}{(Qx)^T(Qx)}.$$

The denominator is $(Qx)^T(Qx) = ||Qx||^2 = ||x||^2 = x^Tx$.

So,

$$R(Q^Tx) \;=\; \frac{(QQ^Tx)^T\Lambda(QQ^Tx)}{(QQ^Tx)^T(QQ^Tx)} \;=\; \frac{x^T\Lambda x}{x^Tx} \;=\; \frac{\lambda_1x_1^2 + \cdots + \lambda_nx_n^2}{x_1^2 + \cdots + x_n^2}.$$

Now, if we call the vector $y = Q^Tx$, then we have

$$R(y) \;=\; \frac{\lambda_1x_1^2 + \cdots + \lambda_nx_n^2}{x_1^2 + \cdots + x_n^2}.$$

Now we use scaling: We can suppose that $x_1^2 + \cdots + x_n^2 = 1$; and so, the values attained by $R(y)$ are the values

$$\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2, \text{ where } x_1^2 + \cdots + x_n^2 = 1.$$

Obviously, these values $R(y)$ lie in $[\lambda_1, \lambda_n]$ (as can be seen by setting $x = (1, 0, 0, ..., 0)$ and $x = (0, ..., 0, 1)$, respectively). The minimium is obviously λ_1, which is the smallest eigenvalue. Thus, we have proved the *Rayleigh Principle*:

$$\min_x R(x) \; = \; \lambda_1.$$

4 Ellipsoids

Here we think about the eigenvalues λ_i in terms of axes of a certain ellipsoid, which is a generalization of an ellipse. For us and ellipsoid centered at the origin will by the set of vectors $x = (x_1, ..., x_n)$ satisfying

$$a_1 x_1^2 + \cdots + a_n x_n^2 \; = \; 1,$$

where

$$0 \; < \; a_1 \leq a_2 \leq \cdots \leq a_n.$$

The points on this ellipsoid furthest from the origin form the major axis (connect these points to form a line segment – that segment is the major axis), and the points on the ellipsoid closest to the origin form the minor axis (here, we are implicitly assuming that we have strict inequality $a_1 < a_2$ and $a_{n-1} < a_n$). So, the points on the ellipsoid that are also on the major axis are

$$(\pm 1/\sqrt{a_1}, 0, 0, ..., 0),$$

and the points on the ellipsoid that are on the minor axis are

$$(0, ..., 0, \pm 1/\sqrt{a_n}).$$

Let us now see what this means in terms of the Rayleigh quotient: From the previous section we have that

$$R(y) \; = \; \frac{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2}{x_1^2 + \cdots + x_n^2}.$$

If we assume A is positive definite (as well as symmetric), then these $\lambda_i > 0$. By scaling, we may restrict our attention to vectors x satisfying

$$\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 \;=\; 1.$$

If we do this, then if we let let x be any of the two vectors on the major axis of this ellipsoid, that is

$$x \;=\; (\pm 1/\sqrt{\lambda_1}, 0, ..., 0),$$

we will have

$$R(y) \;=\; \lambda_1,$$

the smallest eigenvalue. Likewise, if we let x be any of the two vectors on the minor axis, that is

$$x \;=\; (0, ..., 0, \pm 1/\sqrt{\lambda_n}),$$

we will have

$$R(y) \;=\; \lambda_n,$$

the largest eigenvalue.

The ellipsoid

$$\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 \;=\; 1$$

has other axes besides the minor and major ones, and these correspond to the other eigenvalues of A.

5 Planar Slices

Suppose we take a planar slice through this ellipsoid, where the plane passes through the origin. How does the Rayleigh quotient vary over this bit of the ellipsoid?

Well, a planar slice where the plane passes through the origin can be described as the set of all vectors x such that

$$x^T z \;=\; 0,$$

where z is a vector perpendicular to this plane. In particular, we would like to know the value of

$$\min_{x^T z = 0} R(x). \qquad (1)$$

This minimum must be at least as large as

$$\min_x R(x) \; = \; \lambda_1.$$

Can we get an upper bound on (1) as well? The answer is 'YES', and perhaps the best way to see this is to work with $R(y)$: If, as before, we have $y = Q^T x$, then

$$\min_{y^T z=0} R(y) \; = \; \min_{x^T(Qz)=0} R(y) \; = \; \min_{x^T(Qz)=0} \frac{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2}{x_1^2 + \cdots + x_n^2}.$$

So, if we let z' be the vector Qz, then we seek

$$\min_{x^T(z')=0} \frac{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2}{x_1^2 + \cdots + x_n^2}.$$

By scaling we can assume that

$$x_1^2 + \cdots + x_n^2 \; = \; 1.$$

If we do this, then we seek

$$\min_{x^T(z')=0} \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2, \text{ subject to } x_1^2 + \cdots + x_n^2 = 1.$$

There is at least one vector of the form

$$x \; = \; (x_1, x_2, 0, 0, ..., 0), \text{ such that } x^T(z') = 0 \text{ and } x_1^2 + x_2^2 = 1.$$

To see this, if $z' = (a_1, ..., a_n)$, then these conditions translate into

$$x_1 a_1 + x_2 a_2 \; = \; 0, \text{ and } x_1^2 + x_2^2 = 1. \tag{2}$$

The first condition gives all points (x_1, x_2) on a line passing through the origin, unless $(a_1, a_2) = (0, 0)$, in which case it consists of all vectors (x_1, x_2). The second condition gives all points (x_1, x_2) on a circle. There clearly are exactly two points (x_1, x_2) that satisfy both of these if $(a_1, a_2) \neq (0, 0)$ (a line crosses the circle in two places). Let (w_1, w_2) be one of these points of intersection. Then, for any vector z

$$\min_{x^T z=0} R(x) \; \leq \; \lambda_1 w_1^2 + \lambda_2 w_2^2 \; \leq \; \lambda_2. \tag{3}$$

6 The Minimax Principle for the Second Largest Eigenvalue

Here we begin with the following basic question: Is there a vector z for which the upper bound of λ_2 in (3) is attained? The answer is yes, as we will see.

If we could force $(w_1, w_2) = (0, 1)$, then we would get this upper bound λ_2. How can we do this? Basically, if we had that $z' = (a_1, ..., a_n) = (1, 0, 0, ..., 0)$ then the only solutions (x_1, x_2) to (2) would be

$$(x_1, x_2) \;=\; (0, \pm 1).$$

Both these solutions give us an $x = (0, \pm 1, 0, ..., 0)$ such that $R(y) = \lambda_2$. But is

$$\min_{x^T(z')=0} R(y) \;=\; \lambda_2 \;?$$

Indeed it is, because

$$x^T(z') = 0 \text{ implies } x = (0, x_2, x_3, ..., x_n),$$

and then if we use scaling we get

$$\begin{aligned}\min_{x^T(z')=0} R(x) \quad &= \min_{\substack{x=(0,x_2,\dots,x_n)\\ x_1^2+x_2^2+\cdots+x_n^2=1}} \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 \\ &= \min_{x_2^2+\cdots+x_n^2=1} \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.\end{aligned}$$

This last minimum is clearly λ_2, and is attained when $x_2 = 1$ and $x_3 = \cdots = x_n = 0$.

So, we must have that

$$\max_{z} \min_{x^T z=0} R(x) \;=\; \lambda_2. \tag{4}$$

To say that $x^T z = 0$ means that x lies in the orthogonal complement of z, which is an $(n-1)$-dimensional subspace. Thus, we may rewrite (4) as

$$\max_{\dim(S)=n-1} \min_{x \in S} R(x) \;=\; \lambda_2.$$

This is saying "λ_2 is the maximum over all $n-1$ dimensional subspaces S of the minimum of all x in that subspace S."

If we generalize the arguments of this section and the previous section further, then we can prove that for all $j = 0, 1, ..., n-1$,

$$\max_{\dim(S)=n-j} \min_{x \in S} R(x) \;=\; \lambda_{j+1}.$$

7 A Dual Version of the Inequality

Now we turn things around. We wish to determine

$$\min_{\dim(S)=j} \max_{x \in S} R(x),$$

which equals

$$\min_{\dim(S')=j} \max_{x \in S'} R(y) \ = \ \min_{\dim(S')=j} \max_{x \in S'} \frac{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2}{x_1^2 + \cdots + x_n^2}.$$

A vector x belongs to S' means that x is orthogonal to some subspace of dimension $n-j$; this subspace T is the orthogonal complement of S', and has a basis $b_1, ..., b_{n-j}$.

We claim that S' contains a vector x of the form $x = (0, 0, ..., 0, a_1, ..., a_{n-j+1})$; that is, S' contains a vector x orthogonal to the $j-1$ basis vectors

$$e_1 = (1, 0, ..., 0), \ e_2 = (0, 1, 0, ..., 0), \ ..., \ e_{j-1} = (0, 0, ..., 1, 0, ..., 0);$$

that is, there exists a vector $x \in \mathbb{R}^n$ which is orthogonal to T and orthogonal to $e_1, ..., e_{j-1}$; that is, there exists a vector $x \in \mathbb{R}^n$ orthogonal to $e_1, ..., e_{j-1}, b_1, ..., b_{n-j}$. Such a vector x must exist since to say that x is orthogonal to these vectors is equivalent to saying that there exists a vector x such that

$$Cx \ = \ 0,$$

where the rows of C are the vectors $e_1, ..., b_{n-j}$. Since C has $n-1$ rows and n columns, a non-zero solution x must exist.

Now, if S' contains such an x, it contains all multiples of x, and in particular, it must contain a muliple of the form $x' = (0, 0, ..., 0, a'_1, ..., a'_{n-j+1})$ satisfying

$$(a'_1)^2 + \cdots + (a'_{n-j+1})^2 \ = \ 1.$$

But then we will have

$$R(y') \ = \ \lambda_j (a'_1)^2 + \cdots + \lambda_n (a'_{n-j+1})^2 \ \geq \ \lambda_j,$$

where $y' = Q^T x'$. So,

$$\max_{x \in S} R(x) \ \geq \ \lambda_j. \tag{5}$$

We now see that there exists a subspace S' for which equality is attained; that is,

$$\max_{x \in S'} R(y) \ = \ \lambda_j.$$

Basically, we take S' to be the subspace generated by $e_1, ..., e_j$. Then, all $x \in S'$ have the form $(a_1, a_2, ..., a_j, 0, ..., 0)$. Consider all such vectors where $a_1^2 + \cdots + a_j^2 = 1$. For any such x we have

$$R(y) \ = \ \lambda_1 a_1^2 + \cdots + \lambda_j a_j^2 \ \leq \ \lambda_j.$$

In fact, this holds for all $x \in S'$, not just those having norm 1. So,

$$\max_{x \in S'} R(y) \ \leq \ \lambda_j,$$

which combined with (5) gives

$$\max_{x \in S'} R(y) \ = \ \lambda_j.$$

It follows that

$$\min_{S_j} \max_{x \in S_j} R(x) \ = \ \lambda_j.$$

8 Examples

Example 1. Suppose that

$$A \ = \ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Note that this matrix is symmetric and positive definite.

The matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$. We also know that

$$\min_{x \in \mathbb{R}^2} R(x) \ = \ \lambda_1 \ = \ -1.$$

Find a vector x for which $R(x) = -1$.

Solution. In Section 3 we determined that by taking $x = (1, 0, 0, ..., 0)$ we minimize $R(y)$. Well, basically, what this is saying is that if x is the eigenvector associated to $\lambda = 1$, then we minimize $R(x)$, and get the value $R(x) = \lambda_1 = 1$. So, we just need to find that eigenvector: We seek x so that

$$0 \;=\; (A+I)x \;=\; \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} x.$$

Clearly, $x = [1 \;\; -1]^T$ is our eigenvector.

Let us check that this indeed gives $R(x) = -1$: We have

$$R(x) \;=\; \frac{x^T(Ax)}{x^Tx} \;=\; \frac{x^T(-x)}{x^Tx} \;=\; -1.$$

Example 2. Suppose that

$$A \;=\; \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

This matrix is symmetric, positive definite, and has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$.

We know that

$$\max_z \min_{x^Tz=0} R(x) \;=\; \lambda_2$$

Find a vector z such that

$$\min_{x^Tz=0} R(x) \;=\; \lambda_2.$$

What is the vector x which achieves this minimum?

Easy Solution. First, we give a solution which does not rely on what we have said in the previous sections (actually, it does rely on what we did previously, although here I will not change from x coordinates to $y = Q^Tx$ coordinates): We observe that since A is diagonalizable, all of $\mathbb{R}^2$ can be described as $a_1v_1 + a_2v_2$, where v_1 and v_2 are the eigenvectors of A. Now suppose we take $z = v_1$. Then, the set of vectors x satisfying $x^Tz = 0$ are those in the orthogonal complement of v_1. Since A is symmetric, we know that the eigenvectors are all orthogonal to each other; and so, the orthogonal complement of subspace spanned by v_1 is the subspace spanned by v_2. So, say $x^T z = 0$ is the same as saying $x = a_2 v_2$, for some scalar a_2. For *any* such vector $x \neq 0$ we will have

$$R(x) \;=\; \frac{x^T(Ax)}{x^T x} \;=\; \frac{x^T(4x)}{x^T x} \;=\; 4 \;=\; \lambda_2.$$

A Solution Using Previous Sections. From the ideas in section 6 we know that if we set $z' = (1,0)$, then $R(y) \geq \lambda_2$ for all x satisfying $x^T z' = 0$; moreover, if we take $x = (0,1)$ then $R(y) = \lambda_2$. Recall that $y = Q^T x$.

As $y = Q^T x$ (so, $x = Qy$), we have that $x^T z' = 0$ is equivalent to saying $y^T(Q^T z') = 0$. Recall that Q^T is the matrix of eigenvectors of A. So, we have that $Q^T z' = Q^T[1\ 0]^T = v_1$, the eigenvector associated to $\lambda_1 = -2$.

What we have then is that if we pick $z = Q^T z' = v_1$, then $R(y) \geq \lambda_2$ for all $y^T z = 0$.

Furthermore, we know that equality here is achieved when $x = (0,1)$; in other words, $y = Q^T x = v_2$, the eigenvector associated to $\lambda_2 = 4$.

Example 3. Suppose that A is the same matrix as in example 1. We know that $R(x)$ assumes all values between $\lambda_1 = -2$ and $\lambda_2 = 4$. So, in particular, there must be a non-zero vector x such that $R(x) = 0$. Find that vector x.

Easy Solution. We know that a vector $x \in \mathbb{R}^2$ can be written as $x = a_1 v_1 + a_2 v_2$. Since v_1 and v_2 are orthogonal, we know that $x^T x = a_1^2||v_1||^2 + a_2^2||v_2||^2$. Now,

$$R(x) \;=\; \frac{x^T(Ax)}{x^T x} \;=\; \frac{x^T(-2a_1 v_1 + 4a_2 v_2)}{a_1^2||v_1||^2 + a_2^2||v_2||^2} \;=\; \frac{-2a_1^2||v_1||^2 + 4a_2^2||v_2||^2}{a_1^2||v_1||^2 + a_2^2||v_2||^2}.$$

To say that $R(x) = 0$ is equivalent to having the numerator vanish; so, we seek a_1 and a_2 so that

$$4a_2^2||v_2||^2 \;=\; 2a_1^2||v_1||^2.$$

That is,

$$\frac{a_2}{a_1} \;=\; \pm\frac{||v_1||}{\sqrt{2}||v_2||}.$$

We just need to know the eigenvectors v_1 and v_2: We know that $(A + 2I)v_1 = 0$; so,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v_1 \;=\; 0.$$

It follows that $v_1 = [1 \ -1]^T$. We also have $(A - 4I)v_2 = 0$; so,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v_2 = 0,$$

which means that $v_2 = [1 \ 1]^T$. So, $||v_1|| = ||v_2|| = \sqrt{2}$, and therefore

$$\frac{a_2}{a_1} = \pm \frac{1}{\sqrt{2}}.$$

If we take $a_1 = \sqrt{2}$ and $a_2 = 1$, then we have

$$x = \sqrt{2}v_1 + v_2 = [(1+\sqrt{2}) \ (1-\sqrt{2})]^T.$$

Let us check: We have that

$$\begin{aligned} x^T A x &= x^T \begin{bmatrix} 4 - 2\sqrt{2} \\ 4 + 2\sqrt{2} \end{bmatrix}^T = (1+\sqrt{2})(4-2\sqrt{2}) + (1-\sqrt{2})(4+2\sqrt{2}) \\ &= 0. \end{aligned}$$