On the Oscillations of Multiplicative Functions Taking Values ± 1

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Abstract

For multiplicative functions $f(n)$, which take on the values ± 1, we show that under certain conditions on $f(n)$, for all x sufficiently large, there are at least $x\exp(-7(\log\log x)\sqrt{\log x})$ values of $n \leq x$ for which $f(n(n+1)) = -1$.

1 Introduction

Given a completely multiplicative function $f(n)$, which takes on the values ± 1, and which has the property that

$$\sum_{n \leq x} f(n) = o(x),$$

one might wonder how often $f(n)$ changes sign. That is, for how many integers $n \leq x$ is $f(n) = -f(n+1)$?

It is a rather simple matter to prove that the number of $n \leq x$ with $f(n) = f(n+1)$ is $\gg x$: By the pigeonhole principle, one can see that there exist $\delta, \delta' = 0, 1$, or 2, with $\delta < \delta'$, such that there are $\gg x$ integers $j \leq x/2 - 1$ with $f(2j+\delta) = f(2j+\delta')$. Then, since $(2j+\delta')/(\delta,\delta') = (2j+\delta)/(\delta,\delta') + 1$, and since $f((2j+\delta')/(\delta,\delta')) = f((2j+\delta)/(\delta,\delta'))$, each such j gives rise to an $n \leq x$ with $f(n) = f(n+1)$.

Despite how similar these two problems seem (counting n's satisfying $f(n) = f(n+1)$, and counting n's satisfying $f(n) = -f(n+1)$), it is an incomparably more difficult problem to show that $f(n) = -f(n+1)$ for $\gg x$ integers $n \leq x$. Even so, there has been some progress on this question in the past

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decade or so. For instance, Harman, Pintz and Wolke (see [2]) proved that for $f(n) = \lambda(n) = (-1)^{\Omega(n)}$ (the Liouville function),

$$\#\{n \leq x \ : \ f(n) = -f(n+1)\} > \frac{x}{\log^{7+o(1)} x}. \tag{1}$$

In [5], A. Hildebrand proved the following bound for general completely multiplicative functions $f(n) = \pm 1$, which gives a much better answer for infinitely many x:

$$\limsup_{x \to \infty} \frac{(\log \log x)^4 \#\{n \leq x \ : \ f(n) = -f(n+1)\}}{x} > 0.$$

Perhaps the methods of Hildebrand can be used to replace the "limsup" with a "liminf", and thereby give a stronger result than our Theorem 1 below.

In this paper, we prove the following result, which generalizes the result of Harman, Pintz, and Wolke (although the lower bound we give is not as sharp as the one they derive for $f(n) = \lambda(n)$):

Theorem 1 *Suppose that $f(n)$ is a completely multiplicative function which takes on the values ± 1, and suppose that*

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) = 0. \tag{2}$$

Then, for x sufficiently large,

$$\#\{n \leq x \ : \ f(n) = -f(n+1)\} \ > \ \frac{x}{L(x)^7}, \tag{3}$$

where $L(x) = \exp((\log \log x)\sqrt{\log x})$.

Remark: We could perhaps improve the $L(x)^7$ to $\exp(C\sqrt{\log x})$ for some constant C; however, improving the $\sqrt{\log x}$ to $(\log x)^\epsilon$ for arbitrary $\epsilon > 0$ seems to require some new ideas. The bottleneck to obtaining such results, using the method in this paper, is the fact that Lemma 7 only holds with $\beta_2 = \exp((\log x)^{1/2+o(1)})$. If one could prove that this Lemma holds for $\beta_1, \beta_2 < \log^B x$, for some $B > 0$, then one could prove an estimate of the same quality as (1), but with 7 replaced with some constant $C > 0$.

The method of proof of the Theorem is apparently new, and proceeds by showing that

$$\#\{n \leq x \ : \ f(n) = -f(\lfloor pn/q \rfloor)\} > \frac{x}{L(x)^6}, \tag{4}$$

where $p, q \leq \exp(10\sqrt{\log x})$ are some pair of primes with $f(p) = f(q)$. For each such n counted, we will have that $f(pn) = -f(q\lfloor pn/q \rfloor)$; and so, since pn and $q\lfloor pn/q \rfloor$ lie in $(pn - q, pn]$, we deduce that this interval contains an integer m with $f(m) = -f(m+1)$.

We show that if (4) fails to hold for all such primes p and q above, then

$$\#\{n \leq x \ : \ f(n) = -f(\lfloor \alpha n \rfloor)\} < \frac{x}{L(x)^3}, \tag{5}$$

for all real numbers α in a certain short (but not too short) interval near 1. Showing that (5) cannot hold, for all such α considered, is a relatively simple task, and can be proved by integrating the left hand side of (5) over all such α, and then showing that this integral cannot be too small.

2 Proof of Theorem 1.

For a given $x > 1$, let $I = I(x)$ denote the interval $[x/(2\beta_1), x/\beta_1]$, where

$$\beta_1 = \beta_1(x) = \exp(10\sqrt{\log x}).$$

For a given positive real number α, let

$$\Sigma(I, \alpha) = \#\{n \in I \ : \ f(n) = -f(\lfloor \alpha n \rfloor)\}.$$

Let $R(x)$ be the set of all rational numbers

$$\alpha = \frac{p_1 p_2 \cdots p_k}{q_1 q_2 \cdots q_k} \in (1/2, 2),$$

where $k < \beta_2 = \beta_2(x) = L(x)^3$, and where $p_1, ..., p_k, q_1, ..., q_k$ are primes $\leq \beta_1$, with $p_i/q_i \in (1/2, 2)$ and $f(p_i) = f(q_i)$. We note that $R(x)$ contains the number 1.

The proof of the Main Theorem will follow from the following two Propositions:

Proposition 2 *If x is sufficiently large, and if there exists $\alpha' \in R(x)$ such that*

$$\Sigma(I, \alpha') \ \geq \ \frac{|I|}{\beta_2}, \tag{6}$$

then (3) holds.

Proposition 3 *For all x sufficiently large, there exists $\gamma \in R(x)$ such that*

$$\Sigma(I,\gamma) > \frac{|I|}{3}.$$

Let γ satisfy the conclusion to this last proposition (we assume x is sufficiently large). Thus, (6) holds for $\alpha' = \gamma$. By Proposition 2, (3) holds, and Theorem 1 is proved. □

3 Proof of Proposition 2.

We will prove the contrapositive of this Proposition. So, suppose that (3) fails to hold. We will show that the hypothesis of the Proposition is false.

In this proof and in later results we will need the following Lemma and its corollaries:

Lemma 4 *If (3) fails to hold (for a particular value of x), and if $\delta(n)$ is any integer-valued function of n satisfying $|\delta(n)| \leq B < x$, then*

$$\sum_{B<n\leq x-B} |f(n+\delta(n)) \ - \ f(n)| \ \leq \ \frac{4Bx}{L(x)^7}. \tag{7}$$

PROOF. For each $n \in (B, x-B]$ where $f(n+\delta(n)) = -f(n)$, the interval $[n, n+\delta(n)]$ (or $[n+\delta(n), n]$ if $\delta(n) < 0$) contains consecutive integers $m, m+1$ such that $f(m) = -f(m+1)$. Thus, the left hand side of (7) is bounded from above by

$$2 \sum_{\substack{m\leq x-1 \\ f(m)=-f(m+1)}} \#\{n \leq x \ : \ |n-m| \leq |\delta(n)|\} \ \leq \ 2 \sum_{\substack{m\leq x-1 \\ f(m)=-f(m+1)}} 2B \ \leq \ \frac{4Bx}{L(x)^7},$$

as claimed. □

An immediate corollary of this Lemma is as follows:

Corollary 5 *If (3) fails to hold, then for any integer $h \geq 1$ and real number $\theta \leq 2$,*

$$\sum_{n\le x/2-h} |f(\lfloor n\theta\rfloor) - f(h\lfloor n\theta/h\rfloor)| \le \frac{2}{\theta}\sum_{m\le x-2h} |f(m) - f(h\lfloor m/h\rfloor)|$$
$$\le \frac{8hx}{\theta L(x)^7}.$$

PROOF. We note that the first inequality of this lemma follows since for each integer $m \ge 1$, there are at most $2/\theta$ integers n such that $\lfloor n\theta\rfloor = m$ (and note that $\lfloor m/h\rfloor = \lfloor\lfloor n\theta\rfloor/h\rfloor = \lfloor n\theta/h\rfloor$); and, the second inequality is an immediate consequence of Lemma 4, since $h\lfloor n/h\rfloor = n + \delta(n)$ where $-h < \delta(n) \le 0$. □

Finally, we have one more corollary:

Corollary 6 *For x sufficiently large, if (3) fails to hold, and if $p, q \le \beta_1$ are primes with $f(p) = f(q)$ and $p/q \in (1/2, 2)$, then*

$$\Sigma(I, p/q) \; < \; \frac{|I|}{\beta_2^2}.$$

PROOF. To prove this corollary we have from the triangle inequality, from the above lemma and corollary, and from the fact that $f(pm) = f(p)f(m) = f(q)f(m) = f(qm)$ for $m \ge 1$,

$$\begin{aligned}
2\Sigma(I, p/q) &= \sum_{n\in I} |f(n) - f(\lfloor pn/q\rfloor)| \\
&\le \sum_{n\in I} |f(n) - f(q\lfloor n/q\rfloor)| \; + \; |f(q\lfloor n/q\rfloor) - f(p\lfloor n/q\rfloor)| \\
&\qquad\quad + |f(\lfloor pn/q\rfloor) - f(p\lfloor n/q\rfloor)| \\
&= \sum_{n\in I} |f(n) - f(q\lfloor n/q\rfloor)| + |f(\lfloor pn/q\rfloor) - f(p\lfloor n/q\rfloor)| \\
&\le \frac{4qx}{L(x)^7} \; + \; \frac{8qx}{L(x)^7} \; < \; \frac{2|I|}{\beta_2^2},
\end{aligned}$$

for x sufficiently large, which proves the corollary. □

Suppose that $\alpha' \in R(x)$ is arbitrary; we may write it as

$$\alpha' = \frac{p_1 p_2 \cdots p_k}{q_1 q_2 \cdots q_k} = \alpha_1\alpha_2\cdots\alpha_k, \text{ where } \alpha_i = \frac{p_i}{q_i} \in (1/2,\ 2),$$

and where $k < \beta_2$ and $p_1, ..., p_k, q_1, ..., q_k < \beta_1$ are primes satisfying $f(p_i) = f(q_i)$. In addition, we may assume that $\alpha_1\alpha_2\cdots\alpha_\ell \in (1/2, 2)$ for all $1 \le \ell \le k$;

that is, through a simple induction argument, one can show that there exists some permutation σ of $1,2,...,k$, such that $\alpha_{\sigma(1)}\alpha_{\sigma(2)}\cdots\alpha_{\sigma(\ell)} \in (1/2,2)$, for $\ell = 1,2,...,k$.

We will show that our assumption that (3) fails to hold implies

$$\Sigma(I,\alpha_1\alpha_2\cdots\alpha_\ell) \;\le\; \frac{|I|\ell}{\beta_2^2}, \text{ for all } 1 \le \ell \le k. \tag{8}$$

Such an inequality, if true, implies that (6) fails to hold (since $\alpha' = \alpha_1\cdots\alpha_k$, where $k < \beta_2$); and so, since $\alpha' \in R(x)$ was arbitrary, we could conclude that (6) fails to hold for all $\alpha' \in R(x)$, which would prove (the contrapositive of) our Proposition.

To see that (8) holds, we first note that it holds for $\ell = 1$, since this follows from Corollary 6. Now suppose that, for proof by induction, (8) holds for $\ell = m < k$. Thus,

$$\sum_{n\in I} |f(n) - f(\lfloor \alpha_1\cdots\alpha_m n\rfloor)| \;=\; 2\Sigma(I,\alpha_1\cdots\alpha_m) \;\le\; \frac{2|I|m}{\beta_2^2}.$$

We will show below that for x sufficiently large, if $m \ge 1$, then

$$\sum_{n\in I} |f(\lfloor \alpha_1\cdots\alpha_m n\rfloor) - f(\lfloor \alpha_1\cdots\alpha_{m+1} n\rfloor)| < \frac{|I|}{\beta_2^2}; \tag{9}$$

and so, from this and our induction hypothesis we will have

$$\begin{aligned} 2\Sigma(I,\alpha_1\cdots\alpha_{m+1}) &= \sum_{n\in I} |f(n) - f(\lfloor \alpha_1\cdots\alpha_{m+1} n\rfloor)| \\ &\le \sum_{n\in I} |f(\lfloor \alpha_1\cdots\alpha_m n\rfloor) - f(\lfloor \alpha_1\cdots\alpha_{m+1} n\rfloor)| \\ &\qquad\qquad + |f(n) - f(\lfloor \alpha_1\cdots\alpha_m n\rfloor)| \\ &\le \frac{|I|}{\beta_2^2} + \frac{2|I|m}{\beta_2^2} \;<\; \frac{2(m+1)|I|}{\beta_2^2}. \end{aligned}$$

which proves the induction step (that is, (8) holds for $\ell = m+1$), and so (8) follows for $1 \le \ell \le k$.

Finally, we have from the triangle inequality and Corollary 5 that for x sufficiently large, the left hand side of (9) is bounded from above by

$$\sum_{n\in I} |f(\lfloor \alpha_1\cdots\alpha_m p_{m+1} n/q_{m+1}\rfloor) - f(p_{m+1}\lfloor \alpha_1\cdots\alpha_m n/q_{m+1}\rfloor)|$$

$$
\begin{aligned}
& + |f(\lfloor \alpha_1 \cdots \alpha_m n \rfloor) - f(p_{m+1} \lfloor \alpha_1 \cdots \alpha_m n / q_{m+1} \rfloor)| \\
\leq\ & \frac{8 q_{m+1} x}{\alpha_1 \cdots \alpha_m L(x)^7} \\
& \quad + \sum_{n \in I} |f(\lfloor \alpha_1 \cdots \alpha_m n \rfloor) - f(q_{m+1} \lfloor \alpha_1 \cdots \alpha_m n / q_{m+1} \rfloor)| \\
\leq\ & \frac{8 q_{m+1} x}{\alpha_1 \cdots \alpha_m L(x)^7} + \frac{8 q_{m+1} x}{\alpha_1 \cdots \alpha_m L(x)^7} \leq \frac{|I|}{\beta_2^2},
\end{aligned}
$$

as claimed. $\square$

4 Proof of Proposition 3.

Let $\beta_1, \beta_2, I, \Sigma(I, \alpha)$ and $R(x)$ be as in the Section 2. We will need the following two results to prove Proposition 3 (which are proved in Section 5):

Lemma 7 *For x sufficiently large and for any pair of real numbers y_1, y_2 satisfying*

$$1 - \frac{1}{\beta_2} < y_1 < y_2 < 1, \text{ where } y_2 - y_1 \geq \frac{1}{x^2},$$

there exists $\theta \in R(x)$ with $\theta \in [y_1, y_2]$.

Lemma 8 *For x sufficiently large there exists a real number $\theta' \in (1 - \beta_2^{-1}, 1 - (2\beta_2)^{-1})$, such that*

$$\Sigma(I, \theta') > \frac{|I|}{3}.$$

To prove Proposition 3, suppose that x is sufficiently large so that we can apply Lemma 8, and let θ' be as appears there. If $\theta' = a/n$ for some integer $n \leq x$, then set $y_1 = \theta'$ and $y_2 = \theta' + 1/x^2$; otherwise, if $\theta' \neq a/n$, for any integer $n \leq x$ (and some integer a), then we let y_1, y_2 be such that $\theta' \in [y_1, y_2]$, $y_2 - y_1 = 1/x^2$, and $[y_1, y_2]$ contains no rationals of the form a/n, $n \leq x$. We claim that for such y_1 and y_2, we will have that for any $\kappa \in [y_1, y_2]$,

$$\lfloor y_1 n \rfloor = \lfloor \theta' n \rfloor = \lfloor \kappa n \rfloor = \lfloor y_2 n \rfloor, \text{ for every } n \in I; \tag{10}$$

and so, for any such κ, this gives

$$\Sigma(I, y_1) = \Sigma(I, \theta') = \Sigma(I, \kappa) = \Sigma(I, y_2). \tag{11}$$

Since $y_2 - y_1 = 1/x^2$, from Lemma 7 we have that there exists $\theta \in R(x) \cap [y_1, y_2]$, and so taking $\kappa = \theta$ in (11), we deduce

$$\Sigma(I, \theta) = \Sigma(I, \theta') < \frac{2|I|}{3}. \qquad \square$$

5 Proofs of Lemmas.

PROOF of Lemma 7.

To prove the Lemma, we will construct a sequence of rational numbers

$$\frac{1}{2} = \alpha_0 < \alpha_1 < \alpha_2 < \cdots \alpha_{t-1} < 1 - \frac{1}{4x^2} < \alpha_t < \alpha_{t+1} = 1, \tag{12}$$

where

$$2 < \frac{1 - \alpha_{i-1}}{1 - \alpha_i} < \frac{\beta_2}{8 \log^2 x} \tag{13}$$

and where each α_i is of the form

$$\alpha_i = \frac{p_1 p_2 \cdots p_s}{q_1 q_2 \cdots q_s} \in R(x), \text{ where } p_j\text{'s and } q_j\text{'s are prime},$$

and where $s < \log x$. It is evident that $t < 4 \log x$ for x sufficiently large, since (12) and (13) give us that

$$4x^2 > \frac{1}{1 - \alpha_{t-1}} > \frac{2}{1 - \alpha_{t-2}} > \cdots > \frac{2^{t-1}}{1 - \alpha_0} = 2^t.$$

If we had such a collection of α_i's, we claim that we could product together (the product can contain repeats) some of them, to produce a close rational approximation to $y_0 = (y_1 + y_2)/2$; moreover, this product will itself lie in $R(x)$. To find such a product, we iterate the following algorithm:

1. Set $j = 0$ and $n_0 = y_0$.

2. Given the real number $n_j \in (1/2, 1)$, we know that there exists $0 \le i \le t$ such that $\alpha_i \le n_j < \alpha_{i+1}$.

3. Set $n_{j+1} = n_j / \alpha_{i+1}$.

4. Set $j \leftarrow j+1$, and repeat step 2 until $1 - 1/(4x^2) \le n_j < 1$.

Let us assume for now that the algorithm terminates. Then, if we let $\theta = \gamma_1 \cdots \gamma_j$ be the product of all the α_i's we divide by each time the algorithm executes step 3, in passing from n_0 to n_j, we will have that

$$1 - \frac{1}{4x^2} \le \frac{n_0}{\theta} = \frac{y_0}{\theta} < 1;$$

and so, since $\theta = \gamma_1 \cdots \gamma_j \le 1$ we will have that $|y_0 - \theta| \le 1/(4x^2)$, which gives us that $\theta \in [y_1, y_2]$. Now, in proving that the algorithm above halts, we will show that $j < t\beta_2/(4\log^2 x)$; and so, if we expand out θ in terms of a ratio of products of primes (by expanding each γ_i as such a product), we will have

$$\theta = \frac{r_1 \cdots r_g}{s_1 \cdots s_g},\ r_i\text{'s and } s_i\text{'s are prime},$$

and $r_i/s_i \in (1/2, 2)$, $f(r_i) = f(s_i)$, and $g < j \log x < \beta_2$. Thus, we will have $\theta \in R(x)$, and the Lemma will follow.

Let us now see that the above algorithm halts with $j < t\beta_2/(4\log^2 x) < \beta_2/\log x$: We will show by induction that if $\alpha_i \le n_0 < \alpha_{i+1}$ (where $i = 0, ..., t$), then the algorithm halts with $j < (t+1-i)\beta_2/(4\log^2 x)$. This clearly holds for $i = t$, since in this case, the above algorithm halts after the first pass, giving $\theta = 1 \in R(x)$. Suppose that, for proof by induction, the algorithm halts with $j < (t+1-i)\beta_2/(4\log^2 x)$ for all n_0 satisyfing $\alpha_i \le n_0 < \alpha_{i+1}$, where $1 \le B \le i \le t$.

Now, If n_0 is such that $\alpha_{B-1} \le n_0 < \alpha_B$, then the algorithm will repeatedly divide by α_B in step 3, say it divides ℓ times, until $n_\ell = n_0/\alpha_B^\ell \ge \alpha_B$. Since the α_i's satisfy (13), we will have that $\ell \le \beta_2/(4\log^2 x)$. We also have that $\alpha_B \le n_\ell \le 1$; and therefore, $\alpha_{i'} \le n_\ell < \alpha_{i'+1}$, where $B \le i' \le t$. By the induction hypothesis, we will have that $j - \ell < (t+1-B)\beta_2/(4\log^2 x)$ (start the algorithm with n_0 equal to this number n_ℓ); so, the algorithm will halt with $j < (t+1-(B-1))\beta_2/(4\log^2 x)$, since $\ell < \beta_2/(4\log^2 x)$, and so the induction step is proved.

To finish the proof of the Lemma, we must prove that the α_i's exist: Suppose that, for proof by induction, we have constructed the numbers $\alpha_0, ..., \alpha_u$. Let $y = (1-\alpha_u)^{-1}(16\log^2 x)^{-1}\beta_2$. We will show that there exists integers $y < m_1 < m_2 < 2y$, such that $1 \le m_2 - m_1 < \beta_2(32\log^2 x)^{-1}$, which will give

$$2 < \frac{1-\alpha_u}{1 - m_1/m_2} < \frac{\beta_2}{8\log^2 x}.$$

Moreover, m_1 and m_2 will each be expressible as

$$m_1 = p_1 \cdots p_s,\ m_2 = q_1 \cdots q_s,\ p_i \text{ and } q_i \text{ prime for } i = 1, ..., s,$$

where $f(p_i) = f(q_i)$ and for $i = 1, 2, ...s$,

$$y^{1/s} < p_i, q_i < y^{1/s}\left(1 + \frac{1}{2s}\right), \text{where } s = \left\lfloor \frac{\sqrt{\log y}}{4} \right\rfloor.$$

One can easily check that for x sufficiently large $s < \sqrt{\log x}$ (this follows from (12) and (13), which give that $(1 - \alpha_u)^{-1} < x^3$), $y^{1/s} < \beta_1/2$, and that $p_i/q_i \in (1/2, 2)$. Thus, $m_1/m_2 \in R(x)$ with $s < \log x$ as claimed; and so, letting $\alpha_{u+1} = m_1/m_2$ will prove the induction step above, giving that the α_i's exist.

We are left to prove that m_1 and m_2 exist: Let $S(y)$ denote the set of all integers which can be written as $m = p_1 \cdots p_s$, where $p_i \in J := [y^{1/s}, y^{1/s}(1 + (2s)^{-1})]$ is prime. We note that all these primes are $\leq \beta_1$, and all such products lie in $[y, 2y]$. Also, by the pigeonhole principle, there exists a subset $T(y) \subseteq S(y)$, with $|T(y)| \geq |S(y)|/s$, and a constant $1 \leq D \leq s$, such that if $m \in T(y)$ has prime factorization $m = p_1 \cdots p_s$, then exactly D of these prime factors p_i (counting repeats) will have $f(p_i) = 1$ (and so, $s - D$ factors p_j will have $f(p_j) = -1$). Thus, for any pair of numbers $m, m' \in T(y)$, we can arrange their prime factorizations so that

$$m = p_1 \cdots p_s,\ m' = q_1 \cdots q_s,\ f(p_i) = f(q_i).$$

Thus, to prove that m_1, m_2 exist, it suffices to show that $|T(y)| > (32y \log^2 x)\beta_2^{-1}$ (since this implies there are at least two elements of $T(y) \subseteq [y, 2y]$ which are at most $\beta_2(32 \log^2 x)^{-1}$ apart): By the Prime Number Theorem, for x sufficiently large there are $> y^{1/s}/(3 \log y)$ primes in J; and so, by some elementary combinatorics and the bounds $s < \sqrt{\log x}$, $n! \leq n^{n-1}$, and $\log y < 3 \log x$, we get

$$\begin{aligned} |T(y)| &> \frac{|S(y)|}{s} > \frac{(y^{1/s}/(3\log y))^s}{s!s} \geq y\left(\frac{1}{3s\log y}\right)^s \\ &> \frac{y}{\exp(2\log\log x\sqrt{\log x})} > \frac{32y\log^2 x}{\beta_2}. \end{aligned} \tag{14}$$

The Lemma now follows. $\square$

PROOF of Lemma 8.

Let $u_0 = 1 - (2\beta_2)^{-1}$, $u_1 = 1 - \beta_2^{-1}$. Since $\Sigma(I,t) \geq 0$ for all t real, to prove the lemma it suffices to show that

$$\int_{u_0}^{u_1} \Sigma(I,t)dt \sim \frac{(u_1 - u_0)|I|}{2}. \tag{15}$$

We have

$$\begin{aligned}\int_{u_0}^{u_1} \Sigma(I,t)dt &= \int_{u_0}^{u_1} \#\{n \in I \ : \ f(n) = -f(\lfloor tn \rfloor)\}dt \\ &= \sum_{n\in I} \sum_{\substack{u_0 n \leq m \leq u_1 n \\ f(m)=-f(n)}} \mu\left(\{t \ : \ m = \lfloor tn \rfloor\}\right),\end{aligned}$$

where μ is the Lebesgue measure.

We have that $m = \lfloor tn \rfloor$ if and only if $m/n \leq t < (m+1)/n$; and so,

$$\mu\left(\{t \ : \ m = \lfloor tn \rfloor\}\right) = \frac{1}{n}.$$

Thus,

$$\int_{u_0}^{u_1} \Sigma(I,t)dt = \sum_{n\in I} \sum_{\substack{u_0 n \leq m \leq u_1 n \\ f(m)=-f(n)}} \frac{1}{n} \tag{16}$$

To estimate this inner sum we use the following result of Hildebrand (see Corollary 1 of [4]):

Theorem 9 *Let $f(n)$ be a real-valued multiplicative function of modulus ≤ 1 satisfying (2), and let $\phi(z)$ satisfy*

$$3 \leq \phi(z) \leq z, \ \log\phi(z) \sim \log z \ (z \to \infty).$$

Then the limit

$$\lim_{z\to\infty} \frac{1}{\phi(z)} \sum_{z-\phi(z)<n<z} f(n) = 0.$$

From this Theorem, with $z = u_1 n$ and $\phi(z) = n(u_1 - u_0)$ we deduce for $x\exp(-\log^{2/3} x) < n < x$

$$\sum_{\substack{u_0 n \leq m \leq u_1 n \\ f(m)=-f(n)}} 1 \sim \frac{n(u_1 - u_0)}{2};$$

and so,

$$\sum_{n\in I}\frac{1}{n}\sum_{\substack{u_0 n\le m\le u_1 n\\ f(m)=-f(n)}} 1 \sim \sum_{n\in I}\frac{1}{n}\,\frac{(u_1-u_0)n}{2} = \frac{(u_1-u_0)|I|}{2}.$$

Putting this into (16), we deduce (15), and our Lemma is proved. □

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