Notes on the Riemann Zeta Function

January 25, 2007

1 The Zeta Function

1.1 Definition and Analyticity

The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ as follows:

$$\zeta(s) \;=\; \sum_{n \geq 1} \frac{1}{n^s}.$$

The fact that this function is analytic in this region of the complex plane is a consequence of the following basic fact:

Theorem 1 *Suppose that $f_1, f_2, ...$ is a sequence of analytic functions on a region D of the complex plane. Further, suppose that this sequence converges uniformly on compact subsets of D to a function $f(s)$. Then, $f(s)$ is analytic on D, and $f_1', f_2', ...$ converges on compact subsets of D to $f'(s)$.*

We apply this theorem with $f(s) = \zeta(s)$ and

$$f_i(s) \;=\; \sum_{n=1}^{i} \frac{1}{n^s}, \text{ and } f_i'(s) \;=\; -\sum_{n=1}^{i} \frac{\log n}{n^s}.$$

The region D will be $\mathrm{Re}(s) > 1$. Any compact subset of D is a subset of some half-plane $H = \{s \;:\; \mathrm{Re}(s) \geq c > 1\}$. It is a fairly easy exercise to show that the sequence $f_1, f_2, ...$ converges uniformly to $f(s) = \zeta(s)$ in H and that $f_1', f_2', ...$ converges to

$$g(s) \;=\; -\sum_{n=1}^{\infty} \frac{\log n}{n^s}.$$

So, by the above theorem, it follows that for $\mathrm{Re}(s) > 1$,

$$\zeta'(s) \;=\; -\sum_{n=1}^{\infty} \frac{\log n}{n^s};$$

that is to say, we can get $\zeta'(s)$ by differentiating the formula above for $\zeta(s)$ term-by-term when $\mathrm{Re}(s) > 1$.

1.2 The Euler Product

As is well known, there is an intimate connection between the zeta function and prime numbers. This connection comes from the Euler product representation for the zeta function given as follows:

$$\zeta(s) \;=\; \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}. \qquad (1)$$

The fact that this infinite product converges is a consequence of the following basic theorem:

Theorem 2 *Suppose that $a_1, a_2, \ldots$ is a sequence of complex numbers, none of which are -1, such that $s_n = a_1 + \cdots + a_n$ converges absolutely. Then, the infinite product*

$$\prod_{i=1}^{\infty} (1 + a_i)$$

converges. [1]

Applying this theorem to our problem, we find that the above infinite product over primes converges, provided

$$\sum_{p \text{ prime}} \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) \qquad (2)$$

converges absolutely. Letting $s = \sigma + it$, we have that

$$\left|\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right| \;\leq\; \frac{1}{p^\sigma}\left(\frac{1}{1 - \frac{1}{p^\sigma}}\right) \;<\; \frac{1}{p^\sigma}\frac{1}{1 - \frac{1}{2^\sigma}}.$$

[1] The condition that none of the a_is be -1 is because in the usual definition of converging infinite products, we don't allow it to 'converge' to 0.

So, the sum on the left-hand-side of (2) converges absolutely and is bounded from above in absolute value by

$$\frac{1}{1-\frac{1}{2^\sigma}} \sum_{p \text{ prime}} \frac{1}{p^\sigma} \;<\; \frac{1}{1-\frac{1}{2^\sigma}} \sum_{n \geq 1} \frac{1}{n^\sigma},$$

which converges for all $\sigma > 1$.

We are left to show that the value of the Euler product is actually $\zeta(s)$. This is a consequence of the fundamental theorem of arithemetic, which says that for every integer $n \geq 1$, there is a unique sequence of non-negative integers $a_2, a_3, a_5, a_7, ...$, one for each prime, such that

$$n \;=\; 2^{a_2} 3^{a_3} 5^{a_5} 7^{a_7} \cdots$$

Note that all but a finite number of these a_i's are zero. Now consider

$$\prod_{\substack{p \text{ prime} \\ p \leq n}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right). \tag{3}$$

Expanding this finite product out into a sum, we get a sum of a bunch of terms of the form

$$\frac{c}{2^{a_2} 3^{a_3} \cdots q^{a_q}},$$

where q is the largest prime less than or equal n. By the fundamental theorem of arithmetic, each such term must have $c = 1$. Also by the fundamental theorem, we must have that the product (3), written as a sum, must include the terms $1 + 1/2^s + 1/3^s + 1/4^s + \cdots + 1/n^s$. So,

$$0 \;<\; \left| \sum_{j=1}^{\infty} \frac{1}{j^s} - \prod_{\substack{p \text{ prime} \\ p \leq n}} \left(1 - \frac{1}{p^s}\right)^{-1} \right| \;<\; \sum_{j>n} \left| \frac{1}{j^s} \right|. \tag{4}$$

Letting n tend to infinity, the right hand side of (4) converges to 0. So, by the squeeze law of limits, we conclude that (1) holds.

Note that this also tells us that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$, because by the definition of a converging infinite product that product does not 'converge' to 0.

1.3 A Basic Meromorphic Continuation to $\mathrm{Re}(s) > 0$

We might try to analytically continue $\zeta(s)$ to a larger region, and there are several things we might try to do this. It turns out that there is a rather nice trick for doing this, at least if we want to get $\zeta(s)$ for $\mathrm{Re}(s) > 0$. The idea is to observe that for $\mathrm{Re}(s) > 1$

$$\left(1 - \frac{2}{2^s}\right)\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{2}{(2n)^s} \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

Thus,

$$\zeta(s) \;=\; \left(1 - 2^{1-s}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

Now, one can show that the infinite sum here converges conditionally so long as $\mathrm{Re}(s) > 0$; moreover, its derivative converges conditionally in the same region.

Notice however that the factor $(1 - 2^{1-s})^{-1}$ has a singularity at $s = 1$, and in fact at $s = 1 + 2\pi ik/\log(2)$, where $k \in \mathbb{Z}$. These poles are all simple. Thus,

$$(1 - 2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

is an analytic function in $\mathrm{Re}(s) > 0$, except possibly at the points $s = 1 + 2\pi ik/\log(2)$, where $k \in \mathbb{Z}$

Actually, it turns out to be the case that the only singuliarity (not counting removable singularities) is at $s = 1$. To see this, we produce another analytic continuation of $\zeta(s)$ similar to the one above: We have that

$$\left(1 - \frac{3}{3^s}\right)\zeta(s) \;=\; 1 + \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s} + \cdots$$

for $\mathrm{Re}(s) > 1$. Now, the sum

$$\sum_{n=1}^{\infty} \frac{\delta(n)}{n^s}, \text{ where } \delta(n) \;=\; \begin{cases} 1, & \text{if} n \equiv 1, 2 \pmod 3; \\ -2, & \text{if } n \equiv 0 \pmod 3). \end{cases}$$

converges conditionally for $\mathrm{Re}(s) > 0$. Thus,

$$(1 - 3^{1-s})^{-1}\left(1 + \frac{1}{2^s} - \frac{2}{3^s} + \cdots\right)$$

is analytic in $\text{Re}(s) > 0$, except possibly at $s = 1 + 2\pi ik/\log(3)$. So, $\zeta(s)$ is analytic in this region, except possibly at the interesection of the sets $\{1 + 2\pi ik/\log 3 \ : \ k \in \mathbb{Z}\}$ and $\{1 + 2\pi ik/\log 2 \ : \ k \in \mathbb{Z}\}$; that is, $\zeta(s)$ is analytic in $\text{Re}(s) > 0$, except possibly at $s = 1$.

It follows that $(s-1)\zeta(s)$ can be analytically continued to all of $\text{Re}(s) > 0$ (where at $s = 1$ it has a removable singularity).

Note also that from this continuation we deduce that in the region $\text{Re}(s) > 0$, $s \neq 1$, $\zeta(s) = \overline{\zeta(\overline{s})}$. In particular, this says that if $s = \sigma + it$ is a root of $\zeta(s)$, then $\overline{s} = \sigma - it$ is also a root.

1.4 Another Continuation

Another way to continue $\zeta(s)$ is to observe that for $\text{Re}(s) > 1$,

$$\zeta(s) \ = \ \int_{1-\epsilon}^{\infty} \frac{d[x]}{x^s},$$

where the integral here is a Stieltjes integral. [2] Thus, integrating by parts, we find that

$$\zeta(s) \ = \ \frac{[x]}{x^s}\bigg|_{1-\epsilon}^{\infty} + s\int_{1-\epsilon}^{\infty} \frac{[x]dx}{x^{s+1}} \ = \ s\int_1^{\infty} \frac{dx}{x^s} - s\int_1^{\infty} \frac{\{x\}dx}{x^{s+1}},$$

where $\{x\} = x - [x]$, which is the fractional part of x. So,

$$\zeta(s) \ = \ \frac{sx^{-s+1}}{1-s}\bigg|_1^{\infty} - s\int_1^{\infty} \frac{\{x\}dx}{x^{s+1}}.$$

For $\text{Re}(s) > 0$ we therefore have that

$$\zeta(s) - \frac{s}{s-1} \ = \ -s\int_1^{\infty} \frac{\{x\}dx}{x^{s+1}}.$$

Notice that the integral on the right hand side here converges to an analytic function for all $\text{Re}(s) > 0$. This then gives and analytic continuation of $f(s) = \zeta(s) - s/(s-1)$ to $\text{Re}(s) > 0$ (with removable singularity at $s = 1$).

[2] The $0 < \epsilon < 1$ here can be taken arbitrarily small. We need this ϵ in order to pick up a jump discontinuity for $[x]$ from $[1-\epsilon] = 0$ to $[1] = 1$.

This therefore tells us not only that $\zeta(s)$ has a simple pole at $s = 1$, but it also tells us its residue at $s = 1$:

$$\mathrm{Res}_{s=1}\zeta(s) \;=\; \mathrm{Res}_{s=1}\frac{s}{s-1} \;=\; 1.$$

This completes our discussion of some of the basic properties of the zeta function.

2 The Functional Equation

2.1 The Mellin Transform

The Mellin transform is a basic integral transform whose main claim to fame is that it helps to transform the symmetries of θ functions to symmetries of ζ functions. It is defined as follows:

$$M(f)(s) \;=\; \int_0^\infty f(t)t^{s-1}dt.$$

It sends a function f defined on the reals to a function $M(f)$ which is analytic in some domain of the complex plane.

2.2 Facts about the Theta Function

The theta function is defined to be

$$\theta(s) \;=\; \sum_{n=-\infty}^{\infty} e^{-n^2\pi s},$$

which is analytic on the right half plane $\mathrm{Re}(s) > 0$. [3] It also satisfies the remarkable functional relation

$$\theta(s^{-1}) \;=\; s^{1/2}\theta(s). \tag{5}$$

[3] Actually, one often defines the theta function with $e^{2\pi i n^2 s}$, instead of $e^{-n^2\pi s}$ as I have written it. These two versions of $\theta(s)$ are the same after an appropriate change of variables; and, in the usual version, we get convergence of the series in the upper half plane, instead of the right half plane in my version.

2.3 The Mellin Transform of $w(s)$

We don't actually take the Mellin transform of $\theta(s)$, but instead take the transform of the related function

$$w(s) \;=\; \sum_{n\geq 1} e^{-\pi n^2 s} \;=\; \frac{\theta(s)-1}{2}. \tag{6}$$

From the functional equation of $\theta(s)$ given in (5), we deduce that for real $s > 0$,

$$2w(1/s) \;=\; \theta(1/s) - 1 \;=\; s^{1/2}\theta(s) - 1 \;=\; 2s^{1/2}w(s) + s^{1/2} - 1. \tag{7}$$

Put another way,

$$w(s) \;=\; s^{-1/2}w(1/s) - \frac{1}{2} + \frac{s^{-1/2}}{2}.$$

Now, then,

$$M(w) \;=\; \int_0^\infty w(t)t^{s-1}dt \;=\; \sum_{n\geq 1}\int_0^\infty e^{-\pi n^2 t}t^{s-1}dt, \tag{8}$$

provided $\mathrm{Re}(s) > 1/2$. Convergence of this first integral is guaranteed because from the functional from (6) we have that for t near 0, $w(1/t) \sim 0$, and therefore from (7),

$$w(t) \;\sim\; t^{-1/2}/2.$$

Thus, $w(t)t^{s-1} \sim t^{s-3/2}/2$ near $t = 0$. The integral from, say, 0 to 1 of this function therefore converges if $\mathrm{Re}(s) > 1/2$. Also, the integral, say, from $t = 1$ to ∞ of $w(t)t^{s-1}$ will be bounded, because near infinity $w(t) \sim e^{-\pi t}$, which decays very rapidly.

The integration of $w(t)t^{s-1}$ in (8) term-by-term is justified because of the rapid convergence of

$$\sum_{n\leq N} e^{-\pi n^2 t}$$

to $w(t)$ (that is, letting $N \to \infty$).

Now, a typical term in infinite series in (8) is

$$\int_0^\infty t^{s-1}e^{-\pi n^2 t}dt \;=\; \frac{1}{\pi^s n^{2s}}\int_0^\infty t^{s-1}e^{-t}dt \;=\; \frac{\Gamma(s)}{\pi^s n^{2s}},$$

where $\Gamma(s)$ is the usual gamma function.

Thus,

$$M(w)(s) \; = \; \pi^{-s}\Gamma(s)\zeta(2s), \tag{9}$$

for $\mathrm{Re}(s) > 1/2$.

On the other hand, $M(w)(s)$ inherits a certain symmetry from the functional equation (7). To see what this symmetry is, we need a trick. The trick is to break the first integral in (8) up into two pieces, one over $t \in (0,1)$ and the other over $t \in (1,\infty)$; and then,

$$\begin{aligned} M(w) \;\; &= \;\; \int_0^1 w(t)t^{s-1}dt \; + \; \int_1^\infty w(t)t^{s-1}dt \\ &= \;\; \int_0^1 t^{s-1}\left(t^{-1/2}w(1/t) - \frac{1}{2} + \frac{t^{-1/2}}{2}\right)dt \\ &\qquad + \int_1^\infty w(t)t^{s-1}dt. \end{aligned} \tag{10}$$

Letting $u = 1/t$, we have that this integral going from $t = 0$ to $t = 1$ becomes

$$\int_1^\infty u^{-1-s}\left(u^{1/2}w(u) - \frac{1}{2} + \frac{u^{1/2}}{2}\right)du \; = \; \int_1^\infty u^{-1/2-s}w(u)du \; - \frac{1}{2s} - \frac{1}{1-2s}$$

So,

$$M(w)(s) \; = \; \int_1^\infty w(t)\left(t^{-1/2-s} + t^{s-1}\right)dt \; - \frac{1}{2s} - \frac{1}{2(1/2-s)}. \tag{11}$$

So, if we make the transformation $s \to 1/2 - s$, we observe that $M(w)(s)$ remains unchanged; in other words,

$$M(w)(s) \; = \; M(w)(1/2 - s). \tag{12}$$

Also, observe that

$$M(w)(s) + \frac{1}{2s} + \frac{1}{2(1/2-s)}$$

is holomorphic, because the integral in (11) is converges for all s. This follows from the bound that for $t \geq 1$,

$$w(t) \; < \; \sum_{n\geq 1} e^{-\pi n t} \; < \; 2e^{-\pi t}.$$

So, $M(w)(s)$ is meromorphic with simple poles at $s = 0$ and $s = 1/2$.

From (12) and (9) we deduce an analytic (meromorphic) continuation of $\zeta(s)$ to the whole complex plane:

$$\pi^{-s}\Gamma(s)\zeta(2s) \;=\; \pi^{s-1/2}\Gamma(1/2-s)\zeta(1-2s), \text{ except at } s = 0, 1/2.$$

In other words,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) \;=\; \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$

except at $s = 0, 1$.

This gives us an analytic (meromorphic) continuation of $\zeta(s)$ to the whole complex plane. The only singularity is at $s = 1$. There are zeros of $\zeta(s)$ when $\mathrm{Re}(s) < 0$ corresponding to the poles of $\Gamma(s/2)$: We have that for any positive integer n that

$$\lim_{s\to -2n} \zeta(-2n) \;=\; \pi^{-1+2n}\Gamma((1+2n)/2)\zeta(2n+1) \lim_{s\to -2n} \Gamma(s/2)^{-1} \;=\; 0.$$

Also note that it makes sense to define the holomorphic function

$$\xi(s) \;=\; \frac{s(1-s)}{2}\pi^{-1/2s}\Gamma(s/2)\zeta(s).$$

This function satisfies

$$\xi(s) \;=\; \xi(1-s).$$

2.4 Zeros of the Zeta Function, and the Riemann Hypothesis

As we have said, $\zeta(s)$ has zeros at the negative even integers, and these are the only such zeros for $\mathrm{Re}(s) < 0$. Also, for $\mathrm{Re}(s) > 1$, $\zeta(s)$ has no zeros. This leaves the region $0 \le \mathrm{Re}(s) \le 1$, which is called the "critical strip".

It turns out that $\zeta(s)$ has no zeros on the lines $\mathrm{Re}(s) = 0$ or 1. And, we already know that if s is a root, then $\overline{s}$ is also a root. The functional equation gives us an additional symmetry, namely that if s is a root, so is $1 - s$.

The famous Riemann Hypothesis asserts that in the critical strip all the roots of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. The truth (or falsity) of this conjecture would have profound consequences concerning the distribution of prime numbers.

3 The Connection with Prime Numbers

It turns out that there is a formula, called the "explicit formula" which relates the zeros of $\zeta(s)$ to the distribution of prime numbers. We will not bother to prove this formula here. To describe the formula, we first define the von Mangoldt function $\Lambda(n)$, which is

$$\Lambda(n) \ = \ \begin{cases} \log p, & \text{if } n = p^a, \ p \text{ prime;} \\ 0, & \text{otherwise.} \end{cases}$$

So, $\Lambda(n)$ is a certain funny weighting on the prime numbers. It is an easy exercise to prove that

$$\log(\text{lcm}(1, 2, 3, 4, ..., n)) \ = \ \sum_{m \leq n} \Lambda(m).$$

It also turns out that for $\text{Re}(s) > 1$, the logarithmic derivative of the zeta function is

$$-\frac{\zeta'(s)}{\zeta(s)} \ = \ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

The explicit formula says that

$$\sum_{n \leq x} \Lambda(n) \ = \ x - \sum_{\substack{\rho : \zeta(\rho) = 0 \\ \text{Re}(\rho) > 0}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}).$$

Actually, this is not quite right: It is true provided that x is not an integer; when x is an integer, the term $\Lambda(x)$ of the above sum should be $\Lambda(x)/2$ – all other terms are correct.

It turns out that there is a finite version of this formula, where we trucate the sum over zeros to those where $|\text{Im}(\rho)| < T$. This formula is

$$\sum_{n \leq x} \Lambda(n) \ = \ x - \sum_{|\text{Im}(\rho)| < T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}) + R(x, T), \tag{13}$$

where

$$|R(x, T)| \ < \ C \frac{x \log^2(xT)}{T} + (\log x) \min\left(1, \frac{x}{T < x >}\right),$$

where $C > 0$ is some constant, and where $< x >$ denote the distance from x to the nearest prime power.

It is also known how many terms there are in the sum over zeros in (13). The number of such zeros is

$$N(T) \;=\; \frac{T}{\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{\pi} + O(\log T).$$

So, if the Riemann Hypothesis were true we could deduce that

$$\sum_{n\le x}\Lambda(n) \;=\; x + O(x^{1/2}\log^2 x) \tag{14}$$

by taking $T = \sqrt{x}$.

It is realatively easy to show that this implies

$$\pi(x) \;=\; \mathrm{Li}(x) + O(x^{1/2}\log x),$$

where

$$\mathrm{Li}(x) \;=\; \int_2^\infty \frac{dt}{\log t} \;\sim\; \frac{x}{\log x},$$

where $\pi(x)$ is the number of primes $\le x$.

Conversely, one can show that if the primes are distributed according to (14), then the Riemann hypothesis is true.