The Simplex Algorithm

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Problem. We seek $x_1, ..., x_n \geq 0$ which minimizes

$$C(x_1, ..., x_n) = c_1 x_1 + \cdots + c_n x_n,$$

subject to the constraint $Ax \geq b$, where A is $m \times n$, $b = m \times 1$.

Through the introduction of m slack variables $w_1, ..., w_m$, we can replace the inequality $Ax \geq b$ with an equality $A'x' \geq b'$, where here A' is $m \times (m+n)$, and $b = m \times 1$. Basically, the variables $w = [w_1 \ \cdots \ w_m]^T$ satisfy

$$w = Ax - b.$$

Our problem then becomes

Problem. Minimize cx, where

$$c = [c_1 \ c_2 \ \cdots \ c_n \ 0 \ 0 \ \cdots \ 0],$$

and

$$x = [x_1 \ \cdots \ x_n \ w_1 \ \cdots w_m],$$

subject to the constraints

$$Ax = b, \text{ and } x \geq 0,$$

where $A = m \times (m+n)$, $b = m \times 1$.

The region of $\mathbb{R}^n$ described by $x_1, ..., x_n$ satisyf-ing these constraints is called the feasible region. The cost cx will be minimized at one of the vertices of the feasible region, and these vertices are where exactly n of the variables $x_1, ..., x_n, w_1, ..., w_m$ are 0, and where the remaining variables are positive.

Actually, it is possible to have vertices where more than n of the variables vanish. This happens if there is some redundancy in our linear system. We will assume that this does not happen for purposes of describing a bare-bones version of the simplex algorithm.

The algorithm has two phases: The first phase locates a vertex of this region, and the second phase finds the vertex where the cost is minimized.

Suppose we have an algorithm for solving phase I. We now see how phase II works. Basically, we hop from vertex to vertex while reducing the cost. So, the method is iterative. Suppose that at the start of one step we are at a vertex v_1, and by the end of that step we reach vertex v_2.

The vertex v_1 will be connected to v_2 by an edge. This edge corresponds to where $n-1$

of the $m + n$ variables are 0 (which in turn corresponds to the intersection of $n-1$ hyperplanes).

Now, at v_1 we will have that n of these variables are 0, and as we pass to v_2, $n-1$ of these will remain 0, while one that was zero will become positive, and one that was positive will become 0.

The variables that are 0 when we are at v_1 we will call free or *out-variables*, and the remaining m variables which are positive we will call basic, or *in-variables*. So, when we hop from v_1 to v_2, exactly one of the in-variables will become an out-variable, and one of the out-variables will become an in-variable.

There are n different edges in the feasible region that are connected to v_1, basically one edge per subset of $n-1$ of the n out-variables (note that there are $\binom{n}{n-1} = n$ such subsets). So, because there are multiple edges, there are multiple choices for which vertex v_2 to hop to from v_1.

How do we decide which vertex to visit next? Basically, we pick a direction along which the cost function $C(x_1, ..., x_n, w_1, ..., w_m)$ decreases. And the heart of the simplex method is coming up with a simple rule for deciding which direction to consider.

To figure out what such a rule is, consider the following example. Suppose we are at a vertex v_1 where $w_1 = w_2 = 0$. Suppose that the cost function is $C = 2w_1 - 3w_2$. Finally, suppose that our constraint $Ax = b$ is

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

So, we have $(x_1, x_2) = (1, 5)$ and $C = 0$. Notice that the left-most 2×2 submatrix of A is the identity matrix, and that x_1 and x_2 are the in-variables, while w_1, w_2 are out-variables.

Now, if hold $w_1 = 0$ while increasing w_2, then the cost will go down. Thus, we know that it is possible to decrease the cost by making w_2 an in-variable. But then, we must decide which of x_1 or x_2 is to become the out-variable.

Basically, we pick the the x_i which will allow us to increase w_2 as much as possible, because that will mean that the cost decreases as much as possible. We are lucky in that our constraints have the special form

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Setting $w_1 = 0$, this is

$$\begin{aligned} x_1 + 4w_2 &= 1, \text{ and} \\ x_2 + w_2 &= 5. \end{aligned}$$

Now, if we decrease x_1 all the way down to 0 (i.e. make it an out-variable), then from the first equation, we can increase w_2 all the way to $1/4$. If we make x_2, then the second equation says that we can let $w_2 = 5$; but if we took $w_2 = 5$, we would violate the first equation, because $x_1 \geq 0$. Thus, the best we can do is $w_2 = 1/4$, so x_1 is the new out variable.

What made the decision easy as far as the vertex to pick next in this example was that:

1. In the matrix A, the part corresponding to the in-variables (or basic variables) formed an identity matrix.

2. The cost function was expressed purely in terms of the out-variables (free variables).

It turns out that we can always arrange for these two things to happen, and perhaps the best way to see this is to form a special matrix, which we will call a tableaux. Our tableaux will be slightly different from the one in your book.

Starting with $Ax = b$, we rearrange variables so that the the in-variables come first. In other words, suppose that, initially x_1 and w_1 were the in-variables, and our equation is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 16 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 21 \end{bmatrix}.$$

Further, suppose we start at the vertex

$$(x_1, x_2, w_1, w_2) \ = \ (1, 0, 1, 0).$$

Then, by moving the variables x_1 and w_1 to the top, we get the system

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 16 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ w_1 \\ x_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 21 \end{bmatrix}.$$

Next, we augment this matrix by adding in the cost function. Say that the cost is $C = x_1 - 15x_2 + 7w_1 + 8w_2$. Then, the augmented system is

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 16 & 6 & 8 \\ 1 & 7 & -15 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ w_1 \\ x_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 21 \\ C \end{bmatrix}.$$

It is best to write this as a single matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 4 \\ 5 & 16 & 6 & 8 & 21 \\ 1 & 7 & -15 & 8 & C \end{bmatrix}.$$

Next, we apply elimination to get a 2×2 identity matrix in the upper left-hand-corner and 0's below identity matrix (in the third row).

We have

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 4 \\ 5 & 16 & 6 & 8 & 21 \\ 1 & 7 & -15 & 8 & C \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 4 \\ 0 & 1 & -4 & 1 & 1 \\ 1 & 7 & -15 & 8 & C \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 14 & 40 & 1 \\ 0 & 1 & -4 & -12 & 1 \\ 1 & 7 & -15 & 8 & C \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 14 & 40 & 1 \\ 0 & 1 & -4 & -12 & 1 \\ 0 & 0 & -1 & 44 & C-8 \end{bmatrix}.$$

Notice that this last line in the last matrix tells us that

$$C = -x_2 + 44w_2 + 8,$$

which is in terms of the out-variables.

So, when we hop to the new vertex, x_2 will become an in-variable, because we can yet decrease C by increasing x_2 away from 0. We leave w_2 as an out-variable, and so it will have value 0. The first two lines of this matrix, together with $w_2 = 0$ imply

$$\begin{aligned} x_1 + 14x_2 &= 1 \\ w_1 - 4x_2 &= 1. \end{aligned}$$

If we were to set $x_1 = 0$, then the first equation would give $x_2 = 1/14$. If we set $w_1 = 0$, then the second equation would give $x_2 = -1/4$, which is impossible. So, it is impossible to have $w_1 = 0$. So, we must have x_1 becomes the new out-variable.

In general, suppose we are at a vertex described by $w_1 = \cdots = w_m = 0$. Further, suppose that by making w_1 become positive we decrease the cost, while keeping $w_2 = \cdots = w_m = 0$ (this corresponds to a choice of an edge). Then, perhaps we have a series of equations of the following type:

$$\begin{aligned} x_1 + 2w_1 &= 1 \\ x_2 + 8w_1 &= 3 \\ x_3 - w_1 &= 5 \\ x_4 + 2w_1 &= 7. \end{aligned}$$

Then, we consider the minimum of the positive ratios 1/2, 3/8, and 7/2 to determine which of $x_1, \ldots, x_4$ will be the new out-variable (free variable). Since 3/8 is the smallest ratio, then x_2 becomes the new out-variable; and so, the new vertex we visit will be where $x_2 = 0$, $w_1 = 3/8$ and $w_2 = \cdots = w_n = 0$.

Three comments:

1. After elimination we could never get an equation of the type $x_1 - 2w_1 = -1$, where the coefficient on w_i and the number on the other side of the equation are both negative. The reason is that $w_1 = \cdots = w_m = 0$ corresponded to the the vertex at which we started the elimination step; however, this would give $x_1 = -1$, which cannot correspond to a vertex point.

2. If all the equations (after elimination and substituting in $w_2 = \cdots = w_m = 0$) are of the form $x_i - \alpha_i w_1 = \beta_i$, where $\alpha_i, \beta_i > 0$, then our feasible region is not bounded, and there is no minimum (we can take w_1 arbitrarily large, and pick the $x_i > 0$ to make all these equations hold; the cost can therefore be made infinitely negative).

3. Suppose that after elimination our cost function was, say,

$$C = x_2 + 44w_2 - 11.$$

Then it would not be possible to decrease the cost by making x_2 or $w_2 > 0$, while setting the other variable to 0. So, we cannot hop to a new vertex while decreasing the cost. There is a good reason for that: In such a case we have found the vertex which minimizes the cost. Thus, the simplex algorithm amounts to iterating the above process of jumping from vertex to vertex until we produce a cost function having all positive coefficients. When this happens, we stop and declare that we have found the minimum cost.

It remains to explain how to solve Phase I.

Suppose we started with the system

$$\begin{aligned} y - x &\geq 1 \\ y + x &\geq 10 \\ -y &\geq -12 \\ x, y &\geq 0. \end{aligned}$$

Then, we add in our slack variables S_1 and S_2, making the system

$$\begin{aligned} y - x - S_1 &= 1 \\ y + x - S_2 &= 10 \\ -y - S_3 &= -12 \\ x, y, S_1, S_2 &\geq 0. \end{aligned}$$

If we were to set $x, y = 0$, in the hopes of finding a vertex, we would have $S_1 = -1$, $S_2 = -10$, and $S_3 = 12$. So, $(x, y, S_1, S_2, S_3) = (0, 0, -1, -10, 12)$ is not a vertex of our (new) feasible region.

The problem was the first two equations involving S_1 and S_2. The idea is to introduce extra variables E_1 and E_2 so that it is easy to spot a vertex of the system. We put these extra variables E_1 and E_2 into the first two equations, and we have

$$\begin{array}{rcl} y - x - S_1 + E_1 & = & 1 \\ y + x - S_2 \ + E_2 & = & 10 \\ -y - S_3 & = & -12 \\ x, y, S_1, S_2, E_1, E_2 & \geq & 0. \end{array}$$

We see that

$$(x, y, S_1, S_2, S_3, E_1, E_2) \ = \ (0, 0, 0, 0, 12, 1, 10)$$

is a vertex. We obtained this in a mindless manner – we set our original variables x, y to 0, and we set $S_1 = S_2$ to 0 because these were the variables we had the problem with before. The values of the remaining variables are then fixed and are positive.

The idea now is to apply linear programming to the above system to the cost function

$$C(x, y, S_1, S_2, S_3, E_1, E_2) = E_1 + E_2.$$

Since we know our original system without the new variables E_1 and E_2 has at least one vertex, there must be a choice for

$$x, y, S_1, S_2, S_3, E_1, E_2 \ \geq \ 0$$

such that $E_1, E_2 = 0$. But this will minimize the cost.

So, minimizing the cost of this new system involving E_1 and E_2 corresponds to finding a vertex in the original system. This gives us our starting vertex in the original system from which we apply Phase II.