An Estimate from Class

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Here I give a writeup of one of the simple estimates we saw in class, namely that if q is some positive integer, then the number of integers $1 \leq n \leq x$ that are coprime to q satisfies

$$\frac{\varphi(q)x}{q} + O(\tau(q)),$$

where $\tau(q)$ is the number of divisors of q.

The proof of this simple fact is via inclusion-exclusion: Given a collection of sets $S_1, ..., S_k$, the number of elements in their union is

$$\sum_{j=1}^{k} |S_j| - \sum_{1 \leq j_1 < j_2 \leq k} |S_{j_1} \cap S_{j_2}| + \sum_{1 \leq j_1 < j_2 < j_3 \leq k} |S_{j_1} \cap S_{j_2} \cap S_{j_3}| - \cdots + (-1)^{k+1} |S_1 \cap \cdots \cap S_k|.$$

So, if some number q has the prime factors $p_1, ..., p_k$, and if we let S_j be the set of all $n \leq x$ that are divisible by p_j, then the set of $n \leq x$ that have a common prime factor with q is $S_1 \cup \cdots \cup S_k$. It follows that the number of integers $n \leq x$ that are coprime to q is

$$x - \sum_{j=1}^{k} |S_j| + \sum_{1 \leq j_1 < j_2 \leq k} |S_{j_1} \cap S_{j_2}| - \cdots + (-1)^k |S_1 \cap \cdots \cap S_k|.$$

We need to estimate the sizes of these intersections: The set $S_{j_1} \cap \cdots \cap S_{j_t}$ is all those $n \leq x$ that are simultaneously divisible by $p_{j_1}, ..., p_{j_t}$; and so,

$$|S_{j_1} \cap \cdots \cap S_{j_t}| \ = \ \left\lfloor \frac{x}{p_{j_1} \cdots p_{j_t}} \right\rfloor.$$

Noting then that

$$(-1)^t \sum_{1\le j_1<\cdots<j_t\le k} |S_{j_1}\cap\cdots\cap S_{j_t}| \ = \ \sum_{\substack{d|p_1\cdots p_k\\ \omega(d)=t}} \mu(d)\lfloor x/d\rfloor,$$

where $\mu(d)$ is the mobius function, we deduce

$$|\{n\le x \ : \ (n,q)=1\}| \ = \ \sum_{d|q}\mu(d)\lfloor x/d\rfloor \ = \ x\sum_{d|q}\frac{\mu(d)}{d} \ + \ O(\tau(q)).$$

This sum over the divisors d of q is a multiplicative function, and can easily be seen to be $\varphi(q)/q$. One way to see this is to note that this sum is $1/q$ times the convolution of μ with the identity function $\iota : m \to m$; we have seen before that this convolution is just $\varphi(q)$, and so our sum is indeed $\varphi(q)/q$.

Putting everything together, we deduce that

$$|\{n\le x \ : \ (n,q)=1\}| \ = \ \frac{x\varphi(q)}{q} \ + \ O(\tau(q)),$$

as claimed.